N65-22580
(ACCESSION NUMBER)
28
(PAGES)
(NASA CR OR TMX OR AD NUMBER)
(THRU)
1
(CODE)
23
(CATEGORY)
FACILITY FORM 602

NASA TT F-8222

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GPO PRICE $
OTS PRICE(S) $
Hard copy (HC) $2.00
Microfiche (MF) .50

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON
May 1962

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[Following is the translation of an article by N. D. Moiseyev in Trudy Gosudarstvennogo Astronomicheskogo Instituta im P. K. Shternberga (Publications of the P. K. Sternberg State Astronomical Institute), Vol.28, 1960, pages 9-23.]

Introduction

The purpose of the present article is to present the basic theory of one particular variant on averaging in the restricted three body problem, one which we call the ortho-interpolatively averaged variant. Its distinguishing feature is that the interpolation functions of the phase coordinates employed in setting up the interpolatively averaged perturbation function appear in the differential equations of the perturbed motion for the simplified problem obtained by such

averaging, and they are integrals of these equations. In order to satisfy this essential condition, the ortho-interpolative condition, certain lines and planes must be orthogonal in certain auxiliary phase spaces. This is how we justify the use of the term for the variant of the interpolatively averaged three body problem considered here. Considering the distinguishing feature of this variant noted above, one could have named it also a closed interpolatively averaged variant.

The above ortho-interpolative or closure condition on the interpolatively averaged restricted three body problem, being a theoretical condition, is naturally not fulfilled rigorously for any small planet or comet. It is therefore proper to speak only of satisfying the condition to a certain degree of precision. Regarding precision, the following can be said. If one considers perturbations of an asteroid's osculating elements by small quantities of first order, then the ortho-interpolative condition will hold out to quantities of first order inclusively if the so called Tisserand criterion holds to the same degree of precision for this asteroid. This criterion is the well-known simplified variant of the Jacobi integral in the restricted three body problem.

But it is known that the Tisserand criterion is more or less well satisfied for quite a number of asteroids and comets of the Jovian group. Therefore the particular variant on

interpolative averaging of the restricted three body problem of interest in the present article is definitely of practical significance.

The research pursued by M. S. Yarov-Yarovy in his dissertation "Interpolative-Analytic Theory of the Motion of Ceres" provides a good concrete example of the application of this theory to the practical calculation of secular and long-period perturbations of the motion of a small planet. Similar work has been started by another student in the Department of Celestial Mechanics of MGU [Moscow State University], Arpad Pal, applying to the small planet Astrea for which the ortho-interpolative conditions are satisfied also with sufficient precision.

In theory and notation used, the present article is closely related to our previous work on the theory of averaged variants of the three body problem, especially to articles (1), (2), (3). [See translators's comment at end.]

For greater clarity we will limit ourselves in the present article to the simplest case of a restricted three body problem, namely to the circular one. The case of the elliptical restricted three body problem may be treated in a completely analogous fashion. In particular, this is done in the above mentioned dissertation of M. S. Yarov-Yarovy.

The choice of phase coordinates to be used in our

theory may be quite diverse. In this article we treat the simplest type of such coordinates, Delaunay's canonical elements. Likewise one may vary the choice of interpolative elements to be used. In the present article we will limit ourselves to only one particular choice of these interpolative elements.

The cited work of Yarov-Yarovy provides an example of a set of interpolative elements different from that discussed here; instead of Delaunay's, he uses Poincare's elements to advantage. We considered a discussion of the two dimensional restricted three body problem superfluous; in the present article will treat only the more general case of the three dimensional circular restricted three body problem,

Section 1. The Differential Equations of Motion

Let us consider the restricted three dimensional circular three body problem, regarding it for definiteness as the problem of the motion of asteroid P under the gravitational attraction of the Sun, S, and of Jupiter, J, these moving according to Kepler's law of circular motion.

Then the differential equations of motion of asteroid P in terms of Delaunay canonical phase coordinates,

$$L = k\sqrt{m_s}\sqrt{a};\ G = k\sqrt{m_s}\sqrt{p};\ H = k\sqrt{m_s}\sqrt{p}\cos i; \qquad (D)$$
$$l = M; \qquad g = \omega; \qquad \bar{\Omega} = \Omega - M_j = \Omega - nt;$$

may be written as follows:

$$\begin{aligned}
\frac{dL}{dt} &= \frac{\partial \Omega}{\partial M};\\
\frac{dG}{dt} &= \frac{\partial \Omega}{\partial \omega};\\
\frac{dH}{dt} &= \frac{\partial \Omega}{\partial \overline{\delta\Omega}};\\
\frac{dM}{dt} &= -\frac{\partial \Omega}{\partial L};\\
\frac{d\omega}{dt} &= -\frac{\partial \Omega}{\partial G};\\
\frac{d\overline{\delta\Omega}}{dt} &= -\frac{\partial \Omega}{\partial H},
\end{aligned} \qquad (1)$$

where the Hamiltonian Ω takes the form

$$\Omega = \frac{k^4 m_s^2}{2L^2} + n_j H + W_j(L, G, H, M, \omega, \overline{\delta\Omega}), \qquad (2)$$

W_j being the perturbation function. In the problem under consideration the perturbation function may be represented by the triple series:

$$W_j = m_j \sum_{k_1=0}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{k_3=-\infty}^{\infty} C_{k_1 k_2 k_3}(L, G, H) \cos [k_1 M + k_2 \omega + k_3 \overline{\delta\Omega}]. \qquad (3)$$

Differential equations (1) have one first integral,

$$\Omega(L, G, H, M, \omega, \overline{\delta\Omega}) = \text{const}, \qquad (4)$$

which is none other than the Jacobi integral.

Section 2. Interpolative Elements

We will now consider the following two linear functions of the Delaunay elements L, H and M, $\overline{\delta\Omega}$:

$$\varepsilon_1 = \varkappa_{11} L + \varkappa_{13} H \qquad (5.1)$$

and

$$\varepsilon_3 = \varkappa_{34} M + \varkappa_{36} \overline{\Omega}, \tag{5.3}$$

where $\varkappa_{lm}$ are constant coefficients different from zero.

In addition we will consider a similar function of the Delaunay elements G and ω :

$$\varepsilon_2 = \varkappa_{22} G + \varkappa_{25} \omega, \tag{5.2}$$

where coefficients $\varkappa_{lm}$ also are regarded as constants.

The quantities ε_1, ε_2, and ε_3 defined by formulas (5) will hereafter be called interpolative elements. We base this terminology on the manner in which the coefficients $\varkappa_{lm}$ in formulas (5) are to be calculated: a statistical process of linear correlation is applied to the data comprising the set of osculating values of the Delaunay elements for the actual asteroid under study at various times of osculation; that is, formulas (5.1), (5.2), and (5.3) are fitted statistically to the numerical data of Table (6) below.

$$\begin{aligned}
&L_1,\ G_1,\ H_1,\ M_1,\ \omega_1,\ \overline{\Omega}_1 \quad at\ t = t_1,\\
&L_2,\ G_2,\ H_2,\ M_2,\ \omega_2,\ \overline{\Omega}_2 \quad at\ t = t_2,\\
&\cdots\cdots\cdots\quad\cdots\cdots\cdots\\
&\cdots\quad\cdots\cdots\cdots\\
&L_N,\ G_N,\ H_N,\ M_N,\ \omega_N,\ \overline{\Omega}_N \quad at\ t = t_N.
\end{aligned} \tag{6}$$

This statistical fit to the data of Table (6) gives the most probable values of the constants

$$\begin{aligned}
&\tilde{\varepsilon}_1,\ \tilde{\varkappa}_{11},\ \tilde{\varkappa}_{13},\\
&\tilde{\varepsilon}_2,\ \tilde{\varkappa}_{22},\ \tilde{\varkappa}_{25},\\
&\tilde{\varepsilon}_3,\ \tilde{\varkappa}_{34},\ \tilde{\varkappa}_{36},
\end{aligned} \tag{7}$$

while formulas

$$\tilde{\varepsilon}_1 = \tilde{\varkappa}_{11} L + \tilde{\varkappa}_{13} H, \tag{8.1}$$

$$\tilde{\varepsilon}_2 = \tilde{\varkappa}_{22} G + \tilde{\varkappa}_{25} \omega, \tag{8.2}$$

$$\tilde{\varepsilon}_3 = \tilde{\varkappa}_{34} M + \tilde{\varkappa}_{36} \tilde{\Omega} \tag{8.3}$$

give the best (in the least squares sense) representation of the linear correlation between corresponding pairs of Delaunay elements over the ranges of values in Table (6).

After this calculation we will be able to form the following linear functions of the Delaunay elements:

$$\varepsilon_1 = \tilde{\varkappa}_{11} L + \tilde{\varkappa}_{13} H, \tag{9.1}$$

$$\varepsilon_2 = \tilde{\varkappa}_{22} G + \tilde{\varkappa}_{25} \omega, \tag{9.2}$$

$$\varepsilon_3 = \tilde{\varkappa}_{34} M + \tilde{\varkappa}_{36} \Omega, \tag{9.3}$$

which will then define the interpolative elements $\varepsilon_1, \varepsilon_2$ and ε_3 in the true sense. The quantities $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$ and $\tilde{\varepsilon}_3$ will be the average values of these interpolative elements over the ranges of Table (6).

The following comments on the interpolative elements defined by formulas (9) may be noted. Each element ε_1 and ε_3 defined by (9.1) and (9.3) is a linear combination of Delaunay canonical elements from a corresponding linear or angular group, since the angular elements M and Ω from which ε_3 is formed are conjugate to the linear elements L and H from which ε_1 is formed. In this sense it may be said that

interpolative element ε_3 is conjugate to interpolative element ε_1, and vice versa. Such interpolative elements were considered by us in article (2), and the theorems developed in that article will apply directly to what we shall say below about the interpolative elements ε_1, and ε_3 of the averaged problem discussed here. In fact these theorems are basic to the later establishment in the present article of closure or ortho-interpolativity in the averaging process of the restricted three body problem.

On the other hand, interpolative element ε_2 defined by formula (9.2) is of a different type. It is a linear combination formed from the two mutually conjugate canonical Delaunay elements G and ω which correspond to different groups of these elements, the linear and the angular, respectively. As will be demonstrated below, the employment of such a mixed or self-conjugate interpolative element in the averaging process brings a special property. One is not obliged to use it as a supplement to the two mutually conjugate pure interpolative elements. When the ortho-interpolative condition is fulfilled, the integrals of the averaged problem which are identified with constancy of the latter two interpolative elements will exist independently of whether or not interpolative element ε_2 be used to generate still another, a third, averaged problem. An averaging of the problem based

on the self-conjugate interpolative element $\mathcal{E}_2$ yields an integral of the problem which is identified with constancy of this same self-conjugate interpolative element. This results solely from the conjugate property of those two canonical elements, ω and G, from which the self-conjugate interpolative element $\mathcal{E}_2$ is formed.

In order for this result to hold it is not essential to assume a linear dependence of the interpolative element $\mathcal{E}_2$ on the canonical elements G and ω. The relationships just described would not be changed if we were to define the self-conjugate interpolative element $\mathcal{E}_2$, instead of by formula (9.2), by the more general formula

$$\mathcal{E}_2 = \mathcal{E}_2(G, \omega), \tag{10}$$

which would establish a more general correlation than linear between elements G and ω over the values in Table (6).

The development of these remarks is a subject for another article. In the present article, which has as its primary objective the examination of the basic properties of the ortho-interpolatively averaged variant of the three body problem, we will not go into details on that subject.

Section 3. A Double Average Based on Two Mutually Conjugate Interpolative Elements. Integrals of the Doubly Averaged Problem

Proceeding to the consideration of averaged variants

of the restricted three dimensional circular three body problem, we shall first construct its double interpolative average based on the two mutually conjugate interpolative elements ε_1 and ε_3.

To do this we solve equations (9.1) and (9.3) for the quantities H and $\overline{\Omega}$, respectively. This gives the formulas

$$H = \frac{1}{\varkappa_{13}}\varepsilon_1 - \frac{\tilde{\varkappa}_{11}}{\varkappa_{13}} L, \tag{11.1}$$

$$\overline{\Omega} = \frac{1}{\varkappa_{36}}\varepsilon_3 - \frac{\tilde{\varkappa}_{34}}{\varkappa_{36}} M. \tag{11.3}$$

We now put these expressions for H and $\overline{\Omega}$ in formula (2) for the Hamiltonian, Ω. This gives us the following variant of this formula:

$$\Omega = \Omega[L, G, H(\varepsilon_1, L), M, \omega, \overline{\Omega}\varepsilon_3, M)]. \tag{12}$$

If one considers elements ε_1 and ε_3 to be defined as functions of the Delaunay elements by formulas (9.1) and (9.3), then formula (12) becomes an exact formula for the Hamiltonian. However, on taking account of the fact that for the value of coefficients $\varkappa$ obtained as above, the quantities ε_1 and ε_3 defined by formulas (9.1) and (9.3) will vary little over the given time interval from the constant values $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_3$, one may affirm that an assumption of constant ε_1 and ε_3 in the Hamiltonian expression (12) during the time interval considered will lead to only a small error.

Keeping this in mind, we form the averaged expression

for the Hamiltonian by means of the formula

$$\Omega = \frac{1}{(\bar{L}-\underline{L})(\bar{M}-\underline{M})} \int_{L=\underline{L}}^{\bar{L}} \int_{M=\underline{M}}^{\bar{M}} \Omega\,[L, G, H(\varepsilon_1, L), M, \omega, \overline{\Omega}(\varepsilon_3, M)]\, dL dM, \tag{13}$$

where $\bar{L}$, $\underline{L}$ and $\bar{M}$, $\underline{M}$ are the maximum [sic: and minimum] values of the Delaunay elements L and M corresponding to the limits of the empirical data of Table (6) for the time interval considered,

$$\underline{t} = t_1 \leqslant t \leqslant t_N = \bar{t}. \tag{14}$$

During the integration in formula (13) the quantities ε_1 and ε_3 are considered invariant, while the quantities L and M are regarded as independent variables of integration. We consider the pair of Delaunay elements G, ω invariant during the integration.

The doubly averaged expression for the Hamiltonian obtained by formula (13) clearly will depend only on the quantities ε_1, ε_3, G and ω, thus:

$$\tilde{\Omega} = \tilde{\Omega}(\varepsilon_1; \varepsilon_3; G; \omega). \tag{15}$$

However, if one recalls that the interpolative elements ε_1 and ε_3 were defined as functions of the Delaunay elements L, H, M, and $\overline{\Omega}$ by formulas (9.1) and (9.3), then one may consider the averaged Hamiltonian (13) dependent on all the Delaunay elements, this dependence deriving from the auxiliary variables ε_1 and ε_3 as defined in (9.1) and (9.3). In other words we will have

$$\tilde{\Omega} = \tilde{\Omega}\,[\varepsilon_1(L, H); \varepsilon_3(M, \overline{\Omega}); G; \omega]. \tag{16}$$

Let us now replace the true expression (2) for the Hamiltonian Ω in differential equations (1), which describe the three-dimensional circular three body problem, by its expression (16), the result of forming the double interpolative average (13) from the two mutually conjugate interpolative elements ε_1 and ε_3, given by (9.1) and (9.3). We thus obtain the following differential equations of perturbed motion in our doubly averaged variant of the original problem:

$$\begin{aligned} \frac{dL}{dt} &= \frac{\partial \widetilde{\Omega}}{\partial \varepsilon_3} \widetilde{\varkappa}_{34}, & \frac{dM}{dt} &= -\frac{\partial \widetilde{\Omega}}{\partial \varepsilon_1} \widetilde{\varkappa}_{11}, \\ \frac{dG}{dt} &= \frac{\partial \widetilde{\Omega}}{\partial \omega}, & \frac{d\omega}{dt} &= -\frac{\partial \widetilde{\Omega}}{\partial G}, \\ \frac{dH}{dt} &= \frac{\partial \widetilde{\Omega}}{\partial \varepsilon_3} \widetilde{\varkappa}_{36}, & \frac{d\overline{\Omega}}{dt} &= -\frac{\partial \widetilde{\Omega}}{\partial \varepsilon_1} \widetilde{\varkappa}_{13}. \end{aligned} \tag{17}$$

These equations yield two first integrals which are consequences of the simplification obtained by double averaging. For, by combining the first and third equations, and also the fourth and sixth, we obtain

$$\widetilde{\varkappa}_{36} L - \widetilde{\varkappa}_{34} H = \text{const}, \tag{18}$$

$$\widetilde{\varkappa}_{13} M - \widetilde{\varkappa}_{11} \overline{\Omega} = \text{const}. \tag{19}$$

In addition, equations (17) will have the Jacobi integral,

$$\widetilde{\Omega} = \text{const}, \tag{20}$$

appropriate to the restricted circular three body problem.

The following remarks may be made concerning integrals (18) and (19).

Integral (18) may be regarded as a consequence of the fact that we performed the interpolative averaging of the Hamiltonian along the line (8.3) in the coordinate plane of M and $\bar{\Omega}$. If one makes the axis of coordinate M coincide with the axis of coordinate L, and the axis of $\bar{\Omega}$ coincide with the axis of H, then the "interpolation line" (8.3) will lie in the same plane as does the integral line (18). Thus these two lines will be perpendicular to each other.

Similarly, integral (19) may be regarded as a consequence of the fact that the Hamiltonian was subjected to interpolative averaging along line (8.1) By picturing a system of coordinates, along the abcissa of which the coordinates L and M are laid out, and along the ordinate of which are the coordinates H and $\bar{\Omega}$, we will obtain two lines, the "interpolative line" (8.1) and the integral line (19). These two lines will be mutually orthogonal.

These remarks represent a well established special case of the theorems in Section 3. of reference (2).

As to the possibility of completing the integration of equations (17) by quadratures, one can say that from the formal standpoint this possibility is entirely assured by

having the values of the three integrals (18), (19), and (20). However this is technically quite difficult, since the complex structure of the perturbation function, W, complicates the elimination of one or another phase coordinate from the equations via the Jacobi integral (20). As is well known, in similar circumstances, e.g. in solving the Delaunay Hill problem, one comes to resort to a variety of technical aids.

Section 4. The Conditions for Ortho-Interpolation. Their Relationship to the Tisserand Criterion.

In general the integrals (18) and (19), which are obtained after doubly averaging the Hamiltonian on the assumption that the quantities (9.1) and (9.3) maintain constant values, will not coincide with the linear relations (8.1) and (8.3), which describe the latter assumption. In itself, this would not be a shortcoming if this were simply a formal hypothesis.

But relations (8.1) and (8.3) are obtained by processing empirical data, and one may and should regard them as descriptions of empirical regularities which on the average and with some precision are subject to the real time dependence of the osculatory elements of the celestial body. Thus the non-identity of integrals (18) and (19) with relations (8.1) and (8.3) indicates the disagreement which is expected to take place between the integrals of our doubly averaged inter-

polative problem and the empirical regularities in the original problem.

It is quite clear that this is tantamount to an admission of dissatisfaction with our doubly interpolatively averaged simplified variant on the original problem.

In order that our doubly averaged interpolative variant on the original problem be satisfactory in the indicated sense it is necessary that it satisfy the conditions of closure, which may be stated as the conditions of coincidence between linear integrals (18) and (19) and the interpolative linear relations (8.1) and (8.3) on which the averaging was based.

However it is easy to see that these conditions combine into one single condition expressible by the equation.

$$\tilde{x}_{11}\tilde{x}_{34} + \tilde{x}_{13}\tilde{x}_{36} = 0. \qquad (21)$$

Now equation (21) may be viewed as the condition that lines (8.1) and (8.3) be mutually orthogonal in the combined coordinate plane of (L, M) and $(H, \tilde{\Omega})$ already mentioned above.

In fact we call the equation (21) the condition for ortho-interpolation or else for closure of our doubly interpolatively averaged problem.

In accord with the above discussion, this condition will be practically satisfactory for the doubly interpolatively averaged variant of the three body problem being considered.

After all that has been said the question naturally arises of to what extent one may count on finding, among the actual asteroids and comets of the Jovian group, those which will satisfy the ortho-interpolative condition (21) with sufficient precision.

We will show that this question is answered affirmatively for all those asteroids and comets of the Jovian group which satisfy the so-called "Tisserand criterion" with sufficient precision.

Actually the majority of the small bodies of the solar system exhibit quite small perturbations of their osculating elements; among these perturbations, of course, a predominant role is played by the perturbations of first order relative to the perturbing mass which are caused by the action of Jupiter, since perturbations from other planets are extremely small in most cases.

If one limits the consideration to perturbations by Jupiter and at first omits the calculation of perturbations arising from the deviation of Jupiter's eccentricity from zero, then such perturbations will (for the small bodies considered) be given by the equations of a restricted circular three body problem. In particular, these perturbations will satisfy an equation of the form of the Jacobi integral (4). If one limits the calculation to perturbations no higher

than first order relative to the mass of Jupiter, the corresponding approximate Jacobi integral will have the form

$$\frac{k^4 m_s^2}{2L^2} + n_j H = \text{const.} \tag{22}$$

which at the same time represents the well-known Tisserand criterion. This criterion is satisfied fairly well by quite a few asteroids and comets of the Jovian group.

If by L_o and H_o one designates the unperturbed initial values of the osculating elements L and H, and by $\delta_1 L$ and $\delta_1 H$ their first order perturbations, then by the above the Tisserand relation (22) may be written thus:

$$\frac{k^4 m_s^2}{2(L_0 + \delta_1 L)^2} + n_j (H_0 + \delta_1 H) = \text{const.} \tag{23}$$

Neglecting terms higher than first order in $\delta_1 L$ and $\delta_1 H$, we obtain from this relation

$$-n_0 \delta_1 L + n_j \delta_1 H = \text{const.} \tag{24}$$

where by

$$n_0 = \frac{k\sqrt{m_s}}{a_0 \sqrt{a_0}} = \frac{k^4 m_s^2}{L_0^3} \tag{25}$$

is designated the unperturbed initial osculating value for the mean motion of the small body. This linear relation between the first order perturbations of elements L and H is equivalent to a similar relation between the osculating elements L and H themselves if one considers, as is appropriate for the degree of precision chosen in this development, that

the following formulas hold:

$$L = L_0 + \delta_1 L;\ H = H_0 + \delta_1 H. \tag{26}$$

Therefore small bodies (asteroids or comets) whose elements satisfy the Tisserand criterion sufficiently precisely will to the same precision satisfy the following linear relation between osculating elements L and H:

$$-n_0 L + n_j H = \text{const.} \tag{27}$$

Comparing this relation (27) with the linear relation (8.1) between the very same elements, which we assume to have been obtained from a statistical process of correlating the empirical data of Table (6), we see that these relations will coincide if we set

$$\tilde{\varkappa}_{11} = -n_0;\ \tilde{\varkappa}_{13} = n_j. \tag{28}$$

These are theoretical values to which the coefficients $\tilde{\varkappa}_{11}$ and $\tilde{\varkappa}_{13}$ will come close enough when they are obtained by a statistical correlation procedure from the empirical data on any small planet which satisfies the Tisserand criterion well enough in the above sense.

We now turn to the theoretical relationships that one may expect between angular elements M and $\bar{\Omega}$.

For the unperturbed values of these elements we have the formulas

$$M^{(0)} = n_0 t + M_0;\ \bar{\Omega}^{(0)} = \Omega_0 - n_j t. \tag{29}$$

Therefore the unperturbed values of these angular elements are connected by the following linear equations:

$$n_j M^{(0)} + n_0 \overline{\Omega}^{(0)} = n_j M_0 + n_0 \Omega_0 = \text{const.} \tag{30}$$

Setting

$$M = M^{(0)} + \delta M; \quad \overline{\Omega} = \overline{\Omega}^{(0)} + \delta \overline{\Omega} \tag{31}$$

and putting $M^{(0)}$ and $\overline{\Omega}^{(0)}$ from these into equation (30) we will have

$$n_j M + n_0 \overline{\Omega} = [n_j M_0 + n_0 \Omega_0] + (n_j \delta M + n_0 \delta \Omega). \tag{32}$$

Strictly speaking, the right side of this equation is not constant. However the variable term

$$(n_j \delta M + n_0 \delta \Omega) \tag{33}$$

added to the constant term in it is of the same order as the perturbing mass, because the constant term on the right by no means approaches zero along with the perturbing mass. Therefore to the order of precision used in the above case, we have the following approximate linear relation between elements M and $\overline{\Omega}$:

$$n_j M + n_0 \overline{\Omega} = \text{const.} \tag{34}$$

Comparing this with the interpolative equation (8.3), we see that the empirical coefficients obtained by statistically correlating the data in Table (6) should be close to the values

$$\tilde{\varkappa}_{35} = n; \quad \tilde{\varkappa}_{36} = n_0, \tag{35}$$

which make equation (34) coincide with equation (8.3).

Substituting the values (28) and (35) of coefficients $\tilde{\varkappa}_{2m}$ into the ortho-interpolative condition (21), we see that it is satisfied precisely.

Thus one draws the conclusion that the latter condition will be sufficiently precisely satisfied in all cases for which the given actual small celestial body (asteroid or comet) satisfies the Tisserand criterion with sufficient precision. This statement implies that we limit our consideration only to those small bodies for which the perturbation of the osculating elements remains sufficiently small during the time interval considered.

In applying the Tisserand criterion to real asteroids or comets, we will of course always encounter some inconsistent quantity or other. It will disagree more, the more significant the perturbation of the elements and correspondingly the more significant the term W_j, the deviation of which from the Jacobi integral led to the formula for the Tisserand criterion. In addition, this inconsistency will be more significant the greater the effect of those perturbations ascribable to the difference of Jupiter's eccentricity from zero, for only by neglecting these perturbations are we strictly speaking allowed to write the Jacobi integral, the

simplified version of which is the Tisserandcriterion.

However that may be, a survey of actual asteroids and comets of the Jovian group shows that, although Tisserand's criterion and consequently also the ortho-interpolative condition are really not strictly fulfilled, there is nevertheless a number of such in the group for which the criterion and condition are fulfilled with relatively little error. It is this fact which provides the basis for considering a survey of the ortho-interpolative case of the averaged variant of the three body problem to have a definitely practical interest.

Section 5. Integration of the Differential Equations of Perturbed Motion in the Ortho-Interpolatively Doubly Averaged Variant of the Restricted Three Body Problem.

We now turn to the problem of integrating equation (17) when the ortho-interpolative condition (21) holds.

From the foregoing one directly deduces that equations (17) will have two linear integrals (18) and (19), which may be written in the following form:

$$\varepsilon_1 = \tilde{\varkappa}_1 L + \tilde{\varkappa}_{13} H = \tilde{\varepsilon}_1, \qquad (36)$$

$$\varepsilon_3 = \tilde{\varkappa}_{34} M + \tilde{\varkappa}_{36} \overline{\Omega} = \tilde{\varepsilon}_3. \qquad (37)$$

As a consequence of formula (15) and of integrals

(36) and (37), which represent the constancy of interpolative elements ϵ_1 and ϵ_3, the problem reduces first to the problem of integrating the set of two separated equations:

$$\frac{dG}{dt} = \frac{\partial \tilde{\Omega}(\tilde{\varepsilon}_1, \tilde{\varepsilon}_3, G, \omega)}{\partial \omega}, \quad \frac{d\omega}{dt} = -\frac{\partial \tilde{\Omega}(\tilde{\varepsilon}_1, \tilde{\varepsilon}_3, G, \omega)}{\partial G}. \tag{38}$$

On integrating this set of two equations for G and ω, of course one may use the still unexploited Jacobi integral,

$$\tilde{\Omega}[\tilde{\varepsilon}_1, \tilde{\varepsilon}_3, G, \omega] = \text{const.} \tag{20 repeat}$$

Without discussing the actual procedure, we will assume that the set of equations (38) has been integrated, giving

$$G = G(t, G_0, \omega_0, \tilde{\varepsilon}_1, \tilde{\varepsilon}_3); \quad \omega = \omega(t, G_0, \omega_0, \tilde{\varepsilon}_1, \tilde{\varepsilon}_3). \tag{39}$$

Subsequently the right members of the remaining four equations of system (17) may be regarded as known functions of the time and may be integrated by separating the variables. It will be sufficient to do this for the first and fourth equations:

$$L = L_0 + \int_{t=t_0}^{t} \frac{\partial \tilde{\Omega}[\tilde{\varepsilon}_1, \tilde{\varepsilon}_3, G(t), \omega(t)]}{\partial \tilde{\varepsilon}_3} \varkappa_{34} dt, \tag{40}$$

$$M = M_0 - \int_{t=t_0}^{t} \frac{\partial \tilde{\Omega}[\tilde{\varepsilon}_1, \tilde{\varepsilon}_3, G(t), \omega(t)]}{\partial \tilde{\varepsilon}_1} \varkappa_{11} dt. \tag{41}$$

Having determined the time dependence of elements L and M by means of these two quadratures, we find the time dependence of elements H and $\bar{\Omega}$ from equations (8.1) and

(8.3). The arbitrary constants in these integrations are most appropriately taken as the corresponding mean (most probable) values of the corresponding elements obtained by statistically processing the empirical data of Table (6).

In conclusion we remark that we availed ourselves of the circular nature of the problem only in so far as the Jacobi integral (20 repeat) was used in integrating the separated sub-system of two equations (38). Also if the sub-system of equations (38) be integrated by some independent method not requiring the use of the Jacobi integral (20 repeat), then the way of solving the doubly averaged problem described in this section will be satisfactory also for the case of an elliptical restricted three body problem. In the latter case it is true that the right sides of equations (38) will clearly depend on the time as well as on elements G and ω, through the mean anomaly of Jupiter; this fact will of course complicate the integration of sub-system (38). But nothing prevents one from using for this integration, for example, the classical method of successive approximations or a solution in series of powers of the perturbing mass.

Section 6. A Triple Average Based on Two Mutually Conjugate and One Self Conjugate Interpolative Elements. Integration of the Triply Averaged Problem

So much difficulty connected with integrating the doubly averaged interpolative problem, derived by using the two mutually conjugate interpolative elements ε_1 and ε_3, just increases our interest in examining a somewhat simpler variant of that problem which by its nature can provide a first approximation to the solution of the previous problem which does not involve such difficulties in the integration of the simplified equations of perturbed motion.

We obtain this further simplified variant of our problem by subjecting the originally strictly defined problem to not two, but three averagings. Two of these averagings will be carried out as above by means of the two mutually conjugate interpolative elements ε_1 and ε_3. The third averaging will be based on the self conjugate element ε_2 defined by formula (9.2).

To this end, in addition to the two substitutions provided by formulas (11.1) and (11.2) [sic] shown at the beginning of Section 3., we shall introduce into the Hamiltonian Ω still another substitution given by the formula

$$G = \frac{1}{\gamma_{22}} \varepsilon_2 - \frac{\gamma_{23}}{\gamma_{22}} \omega, \qquad (11.2)$$

which is obtained by solving equation (9.2) for G.

Having done this, we define the triply averaged expression for the Hamiltonian, instead of by formula (13), by the similar formula:

$$\bar{\Omega} = \frac{1}{(\bar{L}-\underline{L})(\bar{M}-\underline{M})(\bar{\omega}-\underline{\omega})} \times \tag{42}$$

$$\times \int_{L=\underline{L}}^{\bar{L}} \int_{M=\underline{M}}^{\bar{M}} \int_{\omega=\underline{\omega}}^{\bar{\omega}} \Omega[L, G(\varepsilon_2, \omega), H(\varepsilon_1, L), M, \omega, \overline{\delta\ell}(\varepsilon_3, M)]\, dL\, dM\, d\omega.$$

The quantities $\bar{\omega}$ and $\underline{\omega}$ are respectively the maximum and minimum values of element ω in Table (6). During the integration elements L, M and ω are considered as independent variables, and the quantities ε_1, ε_2 and ε_3 as constants.

The triply interpolatively averaged expression for the Hamiltonian $\tilde{\Omega}$ provided by formula (42) will obviously depend only on the three interpolative elements ε_1, ε_2 and ε_3:

$$\tilde{\Omega} = \tilde{\Omega}[\varepsilon_1, \varepsilon_2, \varepsilon_3]. \tag{43}$$

However, if we are going to regard these interpolative elements as intermediate variables defined by formulas (9.1), (9.2) and (9.3), then the averaged Hamiltonian will finally depend on all six of the Delaunay elements, but via the three intermediate variables ε_1, ε_2 and ε_3 ;

$$\tilde{\Omega} = \tilde{\Omega}[\varepsilon_1(L, H);\ \varepsilon_2(G, \omega);\ \varepsilon_3(M, \overline{\delta\ell})]. \tag{44}$$

The differential equations of such a triply averaged problem will have the form of equations (17) with the difference that the second and fifth equations of that system will be replaced by the following equations:

$$\frac{dG}{dt} = \frac{\partial \widetilde{\Omega}}{\partial \varepsilon_2} \widetilde{\varkappa}_{25}; \quad \frac{d\omega}{dt} = -\frac{\partial \widetilde{\Omega}}{\partial \varepsilon_2} \widetilde{\varkappa}_{22}. \tag{45}$$

The latter sub-set of two equations will have one linear integral:

$$\widetilde{\varkappa}_{22} G + \widetilde{\varkappa}_{25} \omega = \text{const}, \tag{56}$$

[sic]

which describes the condition for constancy of the self-conjugate interpolative element ε_2. One naturally chooses the constant in this integral according to the results of the statistical analysis of Table (6). This makes it possible to write this integral in the form (8.2). Besides this integral the system of equations will have two more integrals of type (18) and (19). If the ortho-interpolative condition obtains, these integrals will take the form of equations (8.1) and (8.3).

Because we know three linear integrals and because of the special form (43) of the Hamiltonian $\widetilde{\Omega}$, we obtain constant quantities [See Note] as the right members of our set of equations. This allows a simple completion of the integration of the differential equations in the present simplified

variant of our problem.

[Note:] (In the elliptical problem these will be known functions of the time.)

[Translator's contribution to references for pages 3 and 13: The articles numbered (1), (2), and (3) are not referenced, but a brief library search indicates that they are most likely the following works on averaged variants ... by Moiseyev.

(1) Vestnik M.G.U., Number 2, 1950, pages 29-

(2) Trudy G.A.I.Sh., Vol.XXIV, 1954, pages 3-9

(3) Ib., pages 10-16]

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